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Abstract

Explicit expressions for the discriminant of Hill's equation up to terms of order four are being calculated. It is shown that these formulas permit a new proof of a theorem about the asymptotic distribution of the characteristic values. They also pèrmit the derivation of certain summation formulas involving the lengths of the intervals of instability and the Fourier Coefficients of the periodic function appearing in Hill's equation.

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1. Introduction

The discriminant of Hill's equation is an analytic funtion $\Delta(\lambda)$ of the characteristic parameter λ appearing in the equation. The zeros of $\Delta - 2$ and of $\Delta + 2$ determine the values λ for which the equation has periodic solutions; the intervals between the zeros of $\Delta^2 - 4$ determine the region of stability and of instability for the solutions of the equation. Therefore, it is of importance for all practical (i.e. numerical) purposes to find expressions for $\Delta(\lambda)$ which may be useful for a computation of the zeros of $\Delta^2 - 4$. Although it is known, in principle, how to find a suitable expansion for $\Delta(\lambda)$, it turns out that the actual computation of the first terms of such an expansion is not an easy task. In Section 4 of this report, the first four terms of the expansion in question are given explicitly. In the Appendix, the various methods for calculating these terms explicitly are discussed and the steps of the computation are indicated.

Apart from being useful for numerical purposes, a discussion of $\Delta(\lambda)$ may also serve for the derivation of general theorems. In Section 4, we shall show how a theorem due to G. Borg[1] can be derived from our expression for Δ, and in Section 6 we shall supplement Borg's result by relations for the zeros of $\Delta - 2$ and of $\Delta + 2$. In the special case where Hill's equation has an even function as its coefficient, we shall be able to prove additional results of the same type, involving the sum of the lengths of the intervals of instability. There, results are connected with a theorem derived by A. Erdélyi[2] who gave an approximate formula for the length of the intervals of instability in the case where the coefficient of Hill's squation is 'almost' constant. We wish

to mention here that Erdélyi's paper[2] also contains some applications of the theory of Hill's equation to the investigation of electric circuits and a large number of references to other applications.

Section 2 contains notations and some elementary results. For a proof of these results we refer to reference [3], where the same notations are being used as in the present report.

The method used in Section 6 for proving relations between the characteristic values is due to Schaefke[4].

2. Notations and elementary results

We shall write Hill's equation in the form

$$y'' + [\lambda + g(x)]y = 0 , \tag{2.1}$$

where λ is a parameter and

$$g(x) = \sum_{n=-\infty}^{\infty} g_n e^{2inx} \tag{2.2}$$

is a real valued periodic function of x with period π; we have

$$g_n = \overline{g}_{-n} \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{2.3}$$

where a bar denotes the conjugate complex of a quantity, and, unless otherwise stated, we shall assume that, for a certain constant $N > 0$,

$$|n^2 g_n| \leqq N . \tag{2.4}$$

Condition (2.4) is satisfied whenever $g(x)$ has a second derivative; it could be weakened without affecting the results. As a normalization, we shall always introduce the condition

$$(2.5) \qquad g_o = 0$$

which is equivalent to

$$\int_0^{\pi} g(x)dx = 0 .$$

By $y_1(x)$ and $y_2(x)$ we denote the solutions of (2.1) which satisfy the initial conditions

$$(2.6) \qquad y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 1 .$$

The discriminant $\Delta(\lambda)$ of Hill's equation can be defined by

$$(2.7) \qquad \Delta = y_1(\pi) + y_2'(\pi) .$$

The transcendental equations in λ:

$$(2.8) \qquad \Delta(\lambda) - 2 = 0$$

and

$$(2.9) \qquad \Delta(\lambda) + 2 = 0$$

have an infinitude of real roots (and no others) which can be arranged in a sequence

$$(2.10) \qquad \lambda_o < \lambda_1' \leqq \lambda_2' < \lambda_1 \leqq \lambda_2 < \lambda_3' \leqq \lambda_4' < \lambda_3 \leqq \lambda_4 \ \ldots$$

where $\lambda_0, \lambda_1, \lambda_2, \ldots$ are the roots of (2.8) and $\lambda_1', \lambda_2', \lambda_3', \ldots$ are the roots of (2.9). If λ equals a root of (2.8), then (2.1) has a non trivial solution (i.e. a solution other than $y \equiv 0$) which is periodic and of period π; if λ satisfies (2.9), then (2.1) has a non trivial solution of period 2π, and there do not exist any other cases where (2.1) has a non trivial solution of period π or 2π.

The intervals $(-\infty, \lambda_0)$, (λ_1', λ_2'), (λ_1, λ_2), (λ_3', λ_4'), (λ_3, λ_4), ... are called <u>intervals of instability</u>. Wherever λ is contained in such an interval, there exists at least one unbounded solution of (2.1). The intervals of instability include their boundary points provided that they contain at least two different points; if $\lambda_1' = \lambda_2'$, the corresponding interval of instability disappears. We shall label the intervals of instability by calling $(-\infty, \lambda_0)$ the zero-th interval, (λ_1', λ_2') the first interval and so on, and, accordingly, we shall talk about <u>even</u> and <u>odd</u> intervals of instability.

The open set of the λ-axis which remains after the intervals of instability have been taken out is a collection of open intervals, the <u>intervals of stability</u>. If λ belongs to such an interval, all solutions of (2.1) are bounded in $(-\infty, \infty)$.

The numbers $\lambda_0, \lambda_1', \lambda_2', \ldots$ are called <u>characteristic values</u> of λ. Whenever two of these coincide, an interval of instability disappears; for the corresponding value of $\lambda = \lambda_{2n-1}' = \lambda_{2n}'$ or $\lambda = \lambda_{2n-1} = \lambda_{2n}$, where $n = 1,2,3,\ldots$, all solutions of (2.1) are periodic with period 2π or π respectively. We shall have $\lambda_{2n-1}' = \lambda_{2n}'$ or $\lambda_{2n-1} = \lambda_{2n}$ if, and only if, (2.7) or (2.8) respectively has a double root; no root of these equations can have a multiplicity greater than 2. For a proof of these facts see [3].

3. Borg's Theorem

The following result is due to G. Borg [1]:

Theorem 1. For large values of $n = 1,2,3,\ldots$ the asymptotic distribution of the characteristic values of Hill's equation is given by

$$\lambda'_{2n-1} = (2n-1)^2 + \frac{C}{(4n-2)^2} + o(n^{-2})$$

$$\lambda'_{2n} = (2n-1)^2 + \frac{C}{(4n-2)^2} + o(n^{-2})$$

$$\lambda_{2n-1} = 4n^2 + \frac{C}{(4n)^2} + o(n^{-2})$$

$$\lambda_{2n} = 4n^2 + \frac{C}{(4n)^2} + o(n^{-2})$$

where

$$C = \frac{1}{\pi}\int_0^{\pi} |g(x)|^2 dx = 2\sum_{n=1}^{\infty} g_n\, g_{-n}\ .$$

We shall prove Theorem 1 fully for the special case where all but a finite number of the g_n are zero. In this case, we may replace $o(n^{-2})$ by $\mathcal{O}(n^{-5/2})$ which shows that the intervals of instability are of the order of magnitude $n^{-5/2}$ as $n \to \infty$. In the general case, we shall indicate how our proof has to be modified in order to obtain Borg's Theorem.

4. Formulas for the discriminant

We shall derive Borg's Theorems from the following two lemmas and from Theorem 3 formulated below:

Lemma 1 The discriminant $\Delta(\lambda)$ is an entire function of λ which admits an expansion

$$\Delta(\lambda) = \sum_{n=0}^{\infty} \Delta_n(\lambda)$$

in which $\Delta_n(\lambda)$ is a homogeneous form in the g_r $(r = \pm 1, \pm 2, \ldots)$ such that the coefficient of

$$g_{r_1} g_{r_2} \cdots g_{r_n}$$

is an entire function of λ of the type

$$R_n(\lambda) \frac{\sin \pi \sqrt{\lambda}}{\sqrt{\lambda}} + S_n(\lambda) \cos \pi \sqrt{\lambda} .$$

There $R_n(\lambda)$, $S_n(\lambda)$ are rational functions of λ the poles of which are located at some of the points

$$\sqrt{\lambda} = \ell_1 + \ell_2 + \ldots + \ell_k$$

where $k \leqq n$ and where $\ell_1, \ldots, \ell_k$ are certain numbers of the set consisting of $r_1, r_2, \ldots, r_n$. For large positive values of λ,

$$\left| \Delta(\lambda) - \sum_{n=0}^{N} \Delta_n(\lambda) \right| = \mathcal{O}(\lambda^{-(N+1)/2}) .$$

Lemma 1 is an immediate consequence of results proved elsewhere; it follows from Theorem 2.5 and from the proof of Theorem 2.4 in [3]. The other lemma needed for the proof of Borg's Theorem will be derived in the Appendix (Section 7). We may formulate it as follows:

<u>Lemma 2.</u> Let $R_n(\lambda)$ and $S_n(\lambda)$ be defined as in Lemma 1. Then the excess of the degree of the denominator over the degree of the numerator is, for $R_n(\lambda)$, at least

$$n - \frac{1}{2}\left[\frac{n}{2}\right] - \frac{1}{2}$$

and for $S_n(\lambda)$ at least

$$n - \frac{1}{2}\left[\frac{n}{2}\right]$$

where $[(n/2)]$ is the largest integer not exceeding $n/2$.

The last result which we shall need for the proof of Borg's Theorem is an explicit expression for Δ_o, Δ_1, Δ_2, Δ_3 and Δ_4. A derivation for these expressions will be given in the Appendix (Section 7). The formulas needed may be written in the following manner:

<u>Theorem 2.</u> <u>For</u> $n \leq 4$, $\Delta_n(\lambda)$ <u>is given by the formulas:</u>

$$\Delta_o(\lambda) = 2\cos\pi\sqrt{\lambda} \quad . \qquad\qquad \Delta_1(\lambda) = 0$$

$$\Delta_2(\lambda) = \frac{\pi\sin\pi\sqrt{\lambda}}{2\sqrt{\lambda}} \sum_{r=1}^{\infty} \frac{g_r g_{-r}}{\lambda - r^2}$$

$$\Delta_3(\lambda) = -\frac{\pi\sin\pi\sqrt{\lambda}}{8\sqrt{\lambda}} \sum_{r,s=1}^{\infty} \frac{(g_r g_s g_{-r-s} + g_{-r} g_{-s} g_{r+s})(3\lambda - r^2 - s^2 - rs)}{(\lambda - r^2)(\lambda - s^2)\left[\lambda - (r+s)^2\right]}$$

$$\Delta_4(\lambda) = -\pi^2 \frac{\cos\pi\sqrt{\lambda}}{16\lambda} \left\{ \sum_{r=1}^{\infty} \frac{g_r g_{-r}}{\lambda - r^2} \right\}^2 - \frac{\pi\sin\pi\sqrt{\lambda}}{64\sqrt{\lambda}} \sum_{r=1}^{\infty} (g_r g_{-r})^2$$

$$\times \frac{30\lambda^2 - 70\lambda r^2 - 16r^4}{(\lambda - r^2)^3(\lambda - 4r^2)} - \frac{\pi\sin\pi\sqrt{\lambda}}{8\sqrt{\lambda}} \sum_{\substack{r,s=1 \\ r > s}}^{\infty} (g_r g_{-r} g_s g_{-s}) \left\{ \frac{5\lambda^2 - 3\lambda(r^2+s^2) + r^2 s^2}{\lambda(\lambda - r^2)^2(\lambda - s^2)^2} \right.$$

$$\left. + \frac{10\lambda - 2r^2 - 2s^2}{(\lambda - s^2)(\lambda - r^2)\left[\lambda - (r-s)^2\right]\left[\lambda - (r+s)^2\right]} \right\} + \sum_{k,\ell,m=1}^{\infty} (G_{k,\ell,m} + G_{-k,-\ell,-m})$$

$$\times S_{k,m,\ell}(\lambda) + \sum_{\substack{k \neq \ell \\ k,\ell=1}}^{\infty} \sum_{m=1}^{\infty} (g_{\ell+m} g_k g_{-\ell} g_{-k-m} + g_{-\ell-m} g_{-k} g_{\ell} g_{k+m}) S_{k,m,\ell,}(\lambda)$$

where

$$G_{k,\ell,m} = g_k g_\ell g_m g_{-k-\ell-m} + g_{k+\ell+m} g_m g_{-k-m} g_{-\ell-m}$$

and $S_{k,m,\ell}(\lambda)$ is defined by

$$4S_{k,\ell,m}(\lambda) = \frac{1}{km}\,\sigma(k+m,\ell) - \frac{1}{k(k+m)}\,\sigma(m,\ell) - \frac{1}{m(k+m)}\,\sigma(k,\ell+m),$$

where

$$\sigma(\ell,m) = \frac{\pi\sin\pi\sqrt{\lambda}}{8\sqrt{\lambda}}\;\frac{(3\lambda-\ell^2-m^2-\ell m)}{(\lambda-\ell^2)(\lambda-m^2)\left[\lambda-(\ell+m)^2\right]}\;.$$

We have embedded in Theorem 3 more information about Δ than would be necessary to prove Borg's Theorems. This has been done since the full explicit expression of Δ_3 and Δ_4 may be expected to be useful for numerical calculations in specific applications.

In the case where $g_{-n} = -g_n$, i.e. where $g(-x) = -g(x)$, we have the

Corollary of Theorem 3: If $g(x)$ is an odd function of x, $\Delta_n \equiv 0$ whenever n is an odd integer.

This result is a trivial by-product of the expression for Δ as an infinite determinant which will be used in the Appendix.

5. Proof of Borg's Theorem

In order to derive Theorem 1 from Lemmas 1 and 2 and from Theorem 2, we shall make the following assumption:

> Only a finite number of the Fourier coefficients g_n of $g(x)$ is different from zero.

This assumption will simplify the proof and will help to clarify the method. We shall indicate afterwards what has to be done if we wish to replace this assumption by a weaker one such as the inequality (2.4). If only finitely many of the g_n are different from zero, it is obvious that, for $\lambda \to \infty$, $\lambda > 0$,

$$\sum_{r=1}^{\infty} \frac{g_r g_{-r}}{\lambda - r^2} = \lambda^{-1} \sum_{r=1}^{\infty} g_r g_{-r} + \mathcal{O}(\lambda^{-2}) \tag{5.1}$$

$$\sum_{\substack{r,s=1 \\ r > s}}^{\infty} g_r g_{-r} g_s g_{-s} \frac{5\lambda^2 - 3\lambda(r^2+s^2) + r^2 s^2}{\lambda(\lambda - r^2)^2 (\lambda - s^2)^2} = \mathcal{O}(\lambda^{-3}) \tag{5.2}$$

$$\sum_{r,s=1}^{\infty} g_r g_{-r} g_s g_{-s} \frac{10\lambda - 2r^2 - 2s^2}{(\lambda - r^2)(\lambda - s^2)\left[\lambda - (r-s)^2\right]\left[\lambda - (r+s)^2\right]} = \mathcal{O}(\lambda^{-3}) \tag{5.3}$$

$$\sum_{r=1}^{\infty} (g_r g_{-r})^2 \frac{30\lambda^2 - 70\lambda r^2 - 16r^4}{(\lambda - r^2)^3 (\lambda - 4r^2)} = \mathcal{O}(\lambda^{-2}) \tag{5.4}$$

(5.5) $$\frac{\sigma(\ell, m)}{\sin\pi\sqrt{\lambda}} = \mathcal{O}(\lambda^{-5/2})$$

(5.6) $$\frac{S_{k,\ell,m}(\lambda)}{\sin\pi\sqrt{\lambda}} = \mathcal{O}(\lambda^{-7/2})$$

Also, it follows from Lemma 2 that

(5.7) $$\Delta_5(\lambda) = (\sin\pi\sqrt{\lambda})\,\mathcal{O}(\lambda^{-7/2}) + (\cos\pi\sqrt{\lambda})\,\mathcal{O}(\lambda^{-4})$$

(5.8) $$\Delta_6(\lambda) = (\sin\pi\sqrt{\lambda})\,\mathcal{O}(\lambda^{-9/2}) + (\cos\pi\sqrt{\lambda})\,\mathcal{O}(\lambda^{-4})$$

and therefore Lemma 1 shows that, for $\lambda \to \infty$,

(5.9) $$\left|\Delta - \sum_{n=0}^{6} \Delta_n\right| = \mathcal{O}(\lambda^{-7/2}) .$$

Consider now the equation

(5.10) $$2 - \Delta = 4\sin^2 \frac{\pi}{2}\sqrt{\lambda} - \Delta_2 - \Delta_3 - \Delta_4 - \dots = 0.$$

If λ is large, all terms except $\sin^2 \frac{\pi}{2}\sqrt{\lambda}$ will be of the order of magnitude of $\lambda^{-3/2}$ or smaller, and therefore we will have

$$\sin \frac{\pi}{2}\sqrt{\lambda^*} = \mathcal{O}(\lambda^{*\,-3/4})$$

if λ^* is a large root of $2 - \Delta = 0$. Therefore,

$$\lambda^* = (2n + \epsilon)^2 , \tag{5.11}$$

where n is an integer and where

$$\epsilon = \mathcal{O}(\lambda^{* \, -3/4}) = \mathcal{O}(n^{-3/2}). \tag{5.12}$$

Since we are interested in λ^* and not in $\sqrt{\lambda^*}$, we may assume that $n > 0$. In terms of ϵ,

$$4\sin^2 \frac{\pi}{2}\sqrt{\lambda^*} = \pi^2\epsilon^2 + \mathcal{O}(\epsilon^4) = \pi^2\epsilon^2 + \mathcal{O}(n^{-6})$$

$$\sin\pi\sqrt{\lambda^*} = \pi\,\epsilon + \mathcal{O}(\epsilon^3) = \pi\,\epsilon + \mathcal{O}(n^{-9/2}) .$$

Therefore, we find from (2.10), by substituting the explicit expressions for Δ_2, Δ_3, Δ_4 from Theorem 3 and by using the inequalities (5.1) to (5.9):

$$\pi^2\left[\epsilon - \frac{1}{32n^3} \sum_{r=1}^{\infty} g_r g_{-r}\right]^2 = \epsilon\,\mathcal{O}(n^{-5}) + \mathcal{O}(n^{-8}). \tag{5.13}$$

From (2.12) we find now

$$\pi^2\left[- \frac{1}{32n^3} \sum_{r=1}^{\infty} g_r g_{-r}\right]^2 = \mathcal{O}(n^{-13/2})$$

and therefore $\epsilon = \mathcal{O}(n^{-3})$. But this implies that, in (2.13)

$$(5.14) \qquad \pi^2 \left[\epsilon - \frac{1}{32n^3} \sum_{r=1}^{\infty} g_r g_{-r} \right]^2 = \mathcal{O}(n^{-8})$$

and therefore we have the result

$$(5.15) \qquad \epsilon = \frac{1}{32n^3} \sum_{r=1}^{\infty} g_r g_{-r} + \mathcal{O}(n^{-4})$$

which leads immediately to

$$(5.16) \qquad \lambda^* = (2n + \epsilon)^2 = 4n^2 + \frac{2}{4n^2} \sum_{r=1}^{\infty} g_r g_{-r} + \mathcal{O}(n^{-3})$$

which agrees with the result stated in Theorem 1. Actually, our remainde term is somewhat smaller than the one mentioned there, but this is due to the fact that (2.1) has to be replaced by a weaker inequality if there ar infinitely many $g_r \neq 0$.

Observe that in (5.15) the coefficient of the leading term n^{-3} cann vanish unless all of the g_r vanish, in which case everything is trivial. Therefore, n^{-3} is the true leading term in the asymptotic expansion for ϵ except for the trivial case $\epsilon \equiv 0$.

Equation (5.16) is not yet the equivalent to the full statement of Theorem 1. What is missing is the proof that (2.14) actually has two roots close to $\sqrt{\lambda} = 2n$ (that these must be real is an elementary result, see [3]), and that these roots are the n-th and the (n+1)-st roots of $2 - \Delta(\lambda) = 0$. To show this, we consider the function

$$D(\lambda, \delta) \;, \qquad\qquad 0 \leq \delta \leq 1$$

which is defined as the discriminant of the equation

$$y'' + [\lambda + \delta g(x)]y = 0 .$$

Clearly, we have

$$D(\lambda,1) = \Delta(\lambda) , \qquad D(\lambda,0) = 2\cos\pi\sqrt{\lambda} .$$

We know that the equation

$$D(\lambda,\delta) - 2 = 0 \qquad (0 \leqq \delta \leqq 1)$$

has real zeros only, and we can show by the method used in proving Lemma 1 that for $\lambda \to \infty$

$$(5.17) \qquad |D(\lambda,\delta) - D(\lambda,0)| \leqq \frac{M}{\lambda} \qquad (M = \text{constant})$$

uniformly in δ. Also, we can show that there exists a negative constant $-\Lambda$ such that

$$(5.18) \qquad |D(-\Lambda,\delta)| > M' \qquad (M' = \text{constant})$$

independent of δ for $0 \leqq \delta \leqq 1$. Now let $\lambda = \alpha + i\beta$ be a complex variable where α, β are real and consider the boundary R of the rectangle defined by

$$-\Lambda \leqq \alpha \leqq (2n + \tfrac{1}{2})^2, \qquad -1 \leqq \beta \leqq 1 .$$

We know from (2.17) and (2.18) that $\Delta(\lambda,\delta) - 2 \neq 0$ on R for $0 \leqq \delta \leqq 1$. We also know that the zeros of the equation

$$\Delta(\lambda,\delta) - 2 = 0$$

which are located within R are continuous functions of δ. Let $N(\delta)$ be their number. Then

$$(5.19) \qquad N(\delta) = \frac{1}{2\pi i} \int_R \frac{\Delta'}{\Delta - 2} d\lambda$$

where $\Delta' = \partial\Delta/\partial\lambda$. Now the integral on the right-hand side is a continuous function of Δ since the denominator never vanishes on the path of integration, and therefore the left-hand side is a constant since it is an integer. For $\delta = 0$, we find

$$N(0) = N(\delta) = 2n + 1 ,$$

and therefore there are exactly $2n + 1$ zeros of $\Delta(\lambda) - 2$ in the interval $- N \leqq \lambda \leqq (2n + \frac{1}{2})^2$. Similarly, there are exactly $2n - 1$ zeros of $\Delta(\lambda) - 2$ in $- N \leqq \lambda = (2n - \frac{3}{2})^2$, and therefore there are exactly two zeros in $2n - 3/2 \leqq \lambda \leqq 2n + 1/2$ which must be the two roots of (5.14) close to $\sqrt{\lambda} = 2n$ and they must satisfy the relation (2.16) for λ^*. This proves Theorem 1 in the case where only finitely many of the g_n are different from zero. In order to indicate what has to be done in general, we shall proceed as follows:

Assume that

$$(5.20) \qquad \sum_{n=1}^{\infty} g_n g_{-n} n^2 < \infty ,$$

i.e. that $g(x)$ has a square integrable first derivative. We wish to estimate the absolute value of

$$D(\lambda) = \sum_{n=1}^{\infty} \left(\frac{g_n g_{-n}}{\lambda - n^2} - \frac{g_n g_{-n}}{\lambda} \right) \sin\pi\sqrt{\lambda}$$

$$= \sum_{n=1}^{\infty} \frac{n^2 g_n g_{-n}}{\lambda(\lambda - n^2)} \sin\pi\sqrt{\lambda}$$

for $\lambda \to \infty$. In the case of a finite sum, we would have $|D(\lambda)| = \mathcal{O}(\lambda^{-2})$. Now all we can do is to prove

$$(5.21) \qquad |D(\lambda)| = \mathcal{O}(\lambda^{-3/2}) \qquad (\lambda \to \infty) .$$

To do this, let $\lambda = \omega^2$ and observe that

$$i \int_{-\pi}^{\pi} e^{i\omega x} n \, g_n g_{-n} \sin nx \, dx = -2(-1)^n \frac{n^2 \sin \omega\pi}{\omega^2 - n^2} g_n g_{-n}$$

and therefore

$$\lambda D(\lambda) = -\frac{i}{2} \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} (-1)^n \, n \, g_n g_{-n} \sin nx \right) e^{i\omega x} \, dx$$

$$= -\frac{i}{2} \int_{-\pi}^{\pi} \frac{d}{dx} \left(\frac{e^{i\omega x} - 1}{i\omega} \right) \left(\sum_{n=1}^{\infty} n \, g_n g_{-n} \sin nx \right) dx \; .$$

Integrating by parts, we find

$$\lambda D(\lambda) = -\frac{1}{2\omega} \int_{-\pi}^{\pi} (e^{i\omega x} - 1) \left(\sum_{n=1}^{\infty} n^2 g_n g_{-n} \cos nx \right) dx \; .$$

Because of (5.20), the integrand is a continuous function of x which is bounded independently of ω. This proves (5.21).

Similar arguments can be applied in order to extend the estimates (5.1) and (5.2) to the case where infinitely many of the g_n are different from zero. However, we shall not go into the details, which are rather tedious.

6. Some relations between the characteristic values.

We shall prove the following results:

Theorem 3. Let the roots of $\Delta(\lambda) + 2$ and of $\Delta(\lambda) - 2$ be denoted as in Section 2. Then

$$\sum_{n=1}^{\infty} \left[\lambda'_{2n-1} + \lambda'_{2n} - 2(2n-1)^2\right] = 0 \tag{6.1}$$

$$\lambda_o + \sum_{n=1}^{\infty} \left[\lambda_{2n-1} + \lambda_{2n} - 2(2n)^2\right] = 0 . \tag{6.2}$$

Whereas Theorem 1 shows that for large n, λ'_{2n-1} and λ'_{2n} exceed $(2n-1)^2$ and λ_{2n-1} and λ_{2n} exceed $(2n)^2$, it follows from Theorem 3 that the same statement cannot be true for all n.

The next result refers to the case where $g(x)$ is an even function of x, i.e. where

$$g(x) = g(-x) \; ; \qquad g_{-n} = g_n \qquad (n = 1,2,3,\dots) .$$

In this case, the periodic solutions of (2.1) belonging to the characteristic values λ'_{2n-1} and λ'_{2n} are either even or odd periodic functions of x with period 2π. Let

$$\gamma'_1 < \gamma'_2 < \dots < \gamma'_n < \dots$$

be the ordered sequence of those numbers of the set λ'_{2n-1}, λ'_{2n} to which there belongs an even periodic function of period 2π, and

$$\sigma'_1 < \sigma'_2 < \dots < \sigma'_n < \dots$$

be the ordered sequence of the remaining numbers of the λ'_{2n-1}, λ'_{2n} to which there belongs an odd periodic solution of (2.1). Similarly, let

$$\gamma_1 < \gamma_2 < \dots < \gamma_n < \dots$$

be the characteristic values belonging to the set of numbers λ_{2n-1} and λ_{2n} to which there belong even solutions of (2.1) which are of period π and

$$\sigma_1 < \sigma_2 < \dots < \sigma_n < \dots$$

the sequence of characteristic values corresponding to odd solutions of period π. Then we have:

<u>Theorem 4:</u>

$$\sum_{n=1}^{\infty} (\gamma'_n - \sigma'_n) = -2 \sum_{n=1}^{\infty} g_{2n-1} \tag{6.3}$$

$$\lambda_o + \sum_{n=1}^{\infty} (\gamma_n - \sigma_n) = -2 \sum_{n=1}^{\infty} g_{2n} \quad . \tag{6.4}$$

The following comment should be made concerning Theorem 4. For large n,

$$\left\{ \begin{aligned} |\gamma'_n - \sigma'_n| &= \lambda'_{2n} - \lambda'_{2n-1} \\ |\gamma_n - \sigma_n| &= \lambda_{2n} - \lambda_{2n-1} \quad . \end{aligned} \right. \tag{6.5}$$

Therefore, the absolute values of the terms in the sums on the left-hand sides of the equations in Theorem 4 represent the lengths of the intervals of instability, at least for large values of n. On the other hand, A. Erdélyi [2] has shown that, for sufficiently small values of

$$\sum_{n=1}^{\infty} |g_n|^2 ,$$

the n-th interval of instability is approximately of length $2|g_n|$. Theorem 4 shows that, although Erdélyi's result may not be exact for the individual intervals of instability, there exist a weaker but exact substitute for it in the form of a relation between sums.

We shall now prove Theorem 3, and we shall confine ourselves to a proof of (6.1). We know from Borg's Theorem (Theorem 1) that

$$(6.6) \qquad (2n - 1)^2 \left[\lambda'_{2n-1} - (2n - 1)^2\right]$$

and

$$(6.7) \qquad (2n - 1)^2 \left[\lambda'_{2n} - (2n - 1)^2\right]$$

are bounded for $n \to \infty$. We also know that $\Delta + 2$ is a function of order of growth $\frac{1}{2}$ and that therefore for a suitable value of the constant c,

$$(6.8) \qquad \Delta + 2 = c \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda'_{2n-1}}\right)\left(1 - \frac{\lambda}{\lambda'_{2n}}\right)$$

unless one of the λ_n' vanishes, in which case we would have to use a slight modification of this formula which will not affect the proof. We know from Theorem 2 that, for large positive values of $\mu = -\lambda$

$$(6.9) \qquad \Delta + 2 = 2\cosh^2 \frac{\pi}{2}\sqrt{\mu}\left[1 - \frac{\pi \sinh \pi\sqrt{\mu}/2}{2\sqrt{\mu}\cosh \pi\sqrt{\mu}/2} \sum_{n=1}^{\infty} \frac{|g_n|^2}{\mu + n^2} + \mathcal{O}(\mu^{-5/2})\right].$$

Now consider the behavior of

$$(6.10) \qquad L(\mu) = \frac{d}{d\mu} \log \frac{\Delta + 2}{\cosh^2(\pi\sqrt{\mu}/2)}$$

for $\mu \to \infty$.

From the product representation of $\Delta + 2$ and of $\cosh(\pi\sqrt{\mu}/2)$ we find

$$(6.11) \qquad L(\mu) = \sum_{n=1}^{\infty} \left\{ \frac{1}{\mu + \lambda_{2n-1}'} + \frac{1}{\mu + \lambda_{2n}'} - \frac{2}{\mu + (2n-1)^2} \right\}$$

$$= -\sum_{n=1}^{\infty} \left\{ \frac{\lambda_{2n-1}' - (2n-1)^2}{(\mu + \lambda_{2n-1}')\left[\mu + (2n-1)^2\right]} + \frac{\lambda_{2n}' - (2n-1)^2}{(\mu + \lambda_{2n}')\left[\mu + (2n-1)^2\right]} \right\}$$

$$= -\sum_{n=1}^{\infty} \left[\lambda_{2n-1}' + \lambda_{2n}' - 2(2n-1)^2\right]\mu^{-2}$$

$$+ \sum_{n=1}^{\infty} \left[\lambda_{2n-1}' - (2n-1)^2\right] \frac{\mu\left[\lambda_{2n-1}' + (2n-1)^2\right] + \lambda_{2n-1}'(2n-1)^2}{\mu^2\left[\mu + (2n-1)^2\right]\left[\mu + \lambda_{2n-1}'\right]}$$

$$+ \sum_{n=1}^{\infty} \left[\lambda'_{2n} - (2n-1)^2\right] \frac{\left[\lambda'_{2n} + (2n-1)^2\right]\mu + \lambda'_{2n}(2n-1)^2}{\mu^2\left[\mu + (2n-1)^2\right]\left[\mu + \lambda'_{2n}\right]} .$$

We wish to show that

$$(6.12) \qquad L(\mu) = - \sum_{n=1}^{\infty} \left[\lambda'_{2n-1} + \lambda'_{2n} - 2(2n-1)^2\right]\mu^{-2} + \mathcal{O}(\mu^{-5/2})$$

for $\mu \to \infty$. For this purpose, we have to estimate the last two sums in (6.11), proving that they are of the order of $\mu^{-5/2}$. Using the boundedness of the expression (6.6) we find that

$$(6.13) \qquad \sum_{n=1}^{\infty} \left[\lambda'_{2n-1} - (2n-1)^2\right] \frac{\mu\left[\lambda'_{2n-1} + (2n-1)^2\right] + \lambda'_{2n-1}(2n-1)^2}{\left[\mu + (2n-1)^2\right]\left[\mu + \lambda'_{2n-1}\right]}$$

can be majorized by

$$S = M \sum_{n=1}^{\infty} \frac{\mu + (2n-1)^2}{\left[\mu + (2n-1)^2\right]^2}$$

where M is a suitable constant. By using a standard procedure we see that S can be written in the form

$$(6.14) \qquad 2M \int_0^{\infty} \frac{\mu + t^2}{(\mu + t^2)^2}\, dt + \mathcal{O}(\mu^{-1}) .$$

Now the integral in (6.14) equals $\pi/2\sqrt{\mu}$ and this proves (6.12).

Next, we shall derive an asymptotic expansion for $L(\mu)$ from (6.9). From some calculations and by using an argument similar to the one we employed in estimating (6.13) we find that

$$L(\mu) = \mathcal{O}(\mu^{-5/2}) \tag{6.15}$$

provided that

$$\sum_{n=1}^{\infty} n^2 |g_n|^2 < \infty \quad .$$

A comparison of (6.12) and (6.15) proves Theorem 3.

The proof of Theorem 4 may be based on the following remarks:

First, we can show that for large n the pair of numbers γ'_n, σ'_n is identical with the pair λ'_{2n-1}, λ'_{2n}, apart from the ordering of the numbers. To prove this statement, we will have to consider the equation

$$y'' + [\lambda + \epsilon g(x)]y = 0 \qquad (0 \leqq \epsilon \leqq 1)$$

using a similar argument as in the proof of Theorem 1. For $\epsilon = 0$, $\lambda'_{2n-1} = \lambda'_{2n} = (2n - 1)^2$. If ϵ increases, the difference $\lambda'_{2n-1} - \lambda'_{2n}$ will, in general, be different from zero, but it will stay small and one of the periodic solutions belonging to these two numbers will be odd, the other, even. Since both λ'_{2n-1} and λ'_{2n} will not leave a certain neighborhood of $(2n - 1)^2$ bounded by, say, $(2n - 1)^2 - \frac{1}{2}$ and $(2n - 1)^2 + \frac{1}{2}$, these two numbers will always be the characteristic values belonging respectively to the n-th even and to the n-th odd periodic solution of period 2π,

although we do not know whether λ_{2n} belongs to an even or to an odd solution.

From this remark and from Borg's Theorem we see that the series (6.3) and (6.4) converge absolutely and can be majorized by the series $M \sum n^{-2}$ where M is a constant. Furthermore, we know that the numbers

$$\gamma'_n, \; \sigma'_n, \; \gamma_n, \; \sigma_n \qquad\qquad (\gamma_o = \lambda_o)$$

are respectively the zeros of

$$y_1(\pi/2,\lambda), \; y'_2(\pi/2,\lambda), y'_1(\pi/2,\lambda), \; y_2(\pi/2,\lambda)$$

where y_1, y_2 are the normalized solutions of (2.1) described in Section 2. (For a proof, see Reference [3] or [4]). From the method of solving (2.1) by iteration, we find that, for $\lambda = -\mu$, μ large and positive

$$(6.16) \qquad y_1(\pi/2,-\mu) = \cosh(\pi\sqrt{\mu}/2)\left[1 - \sum_{n=1}^{\infty} \frac{g_{2n+1}}{\mu + (2n+1)^2} + \mathcal{O}(\mu^{-3/2})\right]$$

$$(6.17) \qquad y_2'(\pi/2,-\mu) = \cosh(\pi\sqrt{\mu}/2)\left[1 + \sum_{n=1}^{\infty} \frac{g_{2n+1}}{\mu + (2n+1)^2} + \mathcal{O}(\mu^{-3/2})\right]$$

$$(6.18) \qquad y_1'(\pi/2,-\mu) = \sqrt{\mu}\,\sinh(\pi\sqrt{\mu}/2)\left[1 - \sum_{n=1}^{\infty} \frac{g_{2n}}{\mu + 4n^2} + \mathcal{O}(\mu^{-3/2})\right]$$

$$(6.19) \qquad y_2(\pi/2,-\mu) = \frac{\sinh \pi\sqrt{\mu}/2}{\sqrt{\mu}} \left[1 + \sum_{n=1}^{\infty} \frac{g_{2n}}{\mu + 4n^2} + \mathcal{O}(\mu^{-3/2}) \right]$$

where the $\mathcal{O}(\mu^{-3/2})$-terms may be differentiated with respect to μ, having a derivative of the order of $\mu^{-5/2}$.

Again, the left-hand sides in (6.16) to (6.19) are functions of μ of order of growth $\frac{1}{2}$, and will admit product representations of the type (6.8), e.g. (for $\lambda_o \neq 0$, $\gamma_n \neq 0$, $\sigma_n \neq 0$)

$$(6.20) \qquad y_1'(\pi/2,-\mu) = c\,(1 + \mu/\lambda_o) \prod_{n=1}^{\infty} (1 + \mu/\gamma_n)$$

$$(6.21) \qquad y_2(\pi/2,-\mu) = c^* \prod_{n=1}^{\infty} (1 + \mu/\sigma_n)$$

where c, c^* are constants. Now we can calculate an asymptotic expansion for

$$\frac{d}{d\mu} \log \frac{y_1'(\pi/2,-\mu)}{y_2(\pi/2,-\mu)}$$

in two different ways, using (6.18) and (6.19) or using (6.20) and (6.21). By equating these two expansions, we find

$$\frac{1}{\mu} + 2\sum_{n=1}^{\infty} \frac{g_{2n}}{\left[\mu + 4n^2\right]^2} + \mathcal{O}(\mu^{-5/2}) = \frac{1}{\mu + \lambda_o} + \sum_{n=1}^{\infty} \frac{\sigma_n - \gamma_n}{(\mu + \gamma_n)(\mu + \sigma_n)}$$

or

$$2 \sum_{n=1}^{\infty} g_{2n} \, \mu^{-2} + \mathcal{O}(\mu^{-5/2}) = - \left[\lambda_o + \sum_{n=1}^{\infty} (\gamma_n - \sigma_n) \right] \mu^{-2} + \mathcal{O}(\mu^{-5/2})$$

which proves (6.4). Equation (6.3) can be proved in the same manner.

Appendix

7. Methods for the computation of Δ

From reference [3], the following results may be taken:

(i) We can define the n-th component Δ_n of

(7.1) $$\Delta = \sum_{n=1}^{\infty} \Delta_n$$

recursively by

(7.2) $$\Delta_n(\lambda) = u_n(\pi,\lambda) + v_n'(\pi,\lambda)$$

where (with the abbreviation $\omega = \sqrt{\lambda}$):

(7.3) $$u_n(x,\lambda) = -\frac{1}{\omega}\int_0^x \sin\omega(x-\xi)g(\xi)u_{n-1}(\xi)d\xi$$

(7.4) $$v_n(x,\lambda) = -\frac{1}{\omega}\int_0^x \sin\omega(x-\xi)g(\xi)v_{n-1}(\xi)d\xi$$

$$u_o = \cos\omega x, \qquad v_o = \omega^{-1}\sin\omega x, \qquad v_n' = dv_n/dx .$$

(ii) An application of the Laplace transformation to (7.2), (7.3) and (7.4) followed by an application of the inversion formula for the Laplace transformation yields the following result: Let

(7.5) $$\Delta_n = \sum_{\ell_1,\ell_2,\ldots\ell_n=-\infty}^{\infty} c(\ell_1,\ldots,\ell_n)g_{\ell_1}g_{\ell_2}\cdots g_{\ell_n} .$$

Then

$$c(\ell_1, \ldots, \ell_n) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} k(\ell_1, \ldots, \ell_n; p) e^{\pi p} dp \tag{7.6}$$

where δ is real and positive and where

$$k(\ell_1, \ldots, \ell_n; p) = (-1)^n \frac{2p - 2i \sum_{v=1}^{n} \ell_v}{\left[\omega^2 + p^2\right] \prod_{v=1}^{n} \left[\omega^2 + (p - 2i\ell_1 - \ldots - 2i\ell_v)^2\right]} \tag{7.7}$$

A consequence of (7.7) is that

$$c(\ell_1, \ldots, \ell_n) = A(\omega) \cos \omega\pi + B(\omega) \sin \omega\pi , \tag{7.8}$$

where $A(\omega)$ and $B(\omega)$ are rational functions of ω the poles of which lie at some or all of the points

$$\omega = 0, \qquad \omega = \pm (\ell_r + \ell_{r+1} + \ldots + \ell_s)$$

where $1 \leq r \leq s \leq n$. The degree of the denominator of $A(\omega)$ and $B(\omega)$ exceeds the degree of the numerator by at least n. $A(\omega)$ and $\omega B(\omega)$ are even functions of ω.

(iii) There exists an expression of Δ in terms of an infinite determinant,

$$2 - \Delta = .4 \sin^2 \frac{\pi}{2} \omega \left| \delta_{n,m} + \frac{g_{n-m}}{\lambda - 4n^2} \right| \tag{7.9}$$

where $\delta_{n,m} = 0$ for $n \neq m$ and $\delta_{m,n} = 1$ and where n,m run from $-\infty$ to $+\infty$ in the infinite determinant on the right-hand side of (7.9) with n denoting the rows and m denoting the columns.

We shall now discuss the merits and disadvantages of the three approaches, (i), (ii) and (iii). The first approach, (i.e. (7.2), (7.3) and (7.4)), provides us with the estimate

$$(7.10) \qquad |\Delta_n| = \mathcal{O}(\omega^{-n}) \qquad \omega \longrightarrow +\infty \; .$$

Also, the same approach permits us to estimate

$$(7.11) \qquad e^{-\pi\theta} \, |\Delta_n|$$

for $\lambda = -\theta^2$, $\theta \to +\infty$. However, (7.2) and (7.3) are not suitable for computing Δ_n if $n > 2$. The reason for this is that (7.2) and (7.3) produce a large number of terms, many of which disappear or combine with others if we take $x = \pi$.

The second approach (i.e. (7.6), (7.7)) provides us with the statements made above about the nature of $A(\omega)$ and $B(\omega)$ in (7.8) if we evaluate the integral in (7.6) by the method of residues. Still, the number of terms to be considered when evaluating this integral is much larger than the number of final terms since many terms amalgamate. The approach described by (7.6) and (7.7) does not allow us to prove Lemma 2 of Section 4. It does not even make evident the fact that only those of the

$$c(\ell_1, \ell_2, \ldots, \ell_n)$$

can be different from zero for which

$$(7.12) \qquad \ell_1 + \ell_2 + \ldots + \ell_n = 0 \; ,$$

although this fact can be derived without any calculations from the remark that the discriminant $\Delta(\lambda)$ must remain unchanged if $g(x)$ is replaced by $g(x - \alpha)$, where α is any real number.

As it happens, the most efficient method of calculating $\Delta(\lambda)$ is still the method originally used by Hill which is based on the expression (7.9) of Δ in terms of an infinite determinant. We shall exemplify this method by applying it to the computation of Δ_2, Δ_3 and Δ_4. In doing this, we shall also indicate how to prove Lemma 2 of Section 4.

If we expand the infinite determinant of formula (7.9) in such a manner that we keep terms together which are of the same degree in the g_n, we find that

$$(7.13) \qquad \Delta_0 = 2 \cos \omega\pi \qquad\qquad \Delta_1 = 0$$

$$(7.14) \qquad \Delta_2 = -4 \sin^2 \frac{\pi}{2}\omega \sum_{\ell=1}^{\infty} \sum_{t=-\infty}^{\infty} \frac{\begin{vmatrix} 0 & g_\ell \\ g_{-\ell} & 0 \end{vmatrix}}{(\lambda - 4t^2)\left[\lambda - 4(t+\ell)^2\right]}$$

$$(7.15) \qquad -\Delta_3 \left[4 \sin^2 \frac{\pi}{2}\omega\right]^{-1}$$

$$= \sum_{\ell,m=1}^{\infty} \sum_{t=-\infty}^{\infty} \frac{\begin{vmatrix} 0 & g_\ell & g_{\ell+m} \\ g_{-\ell} & 0 & g_m \\ g_{-\ell-m} & g_{-m} & 0 \end{vmatrix}}{(\lambda - 4t^2)\left[\lambda - 4(t+m)^2\right]\left[\lambda - 4(t+\ell+m)^2\right]}$$

$$(7.16) \qquad -\Delta_4 \left[4 \sin^2 \frac{\pi}{2} \omega\right]^{-1}$$

$$= \sum_{\ell,m,k=1}^{\infty} \sum_{t=-\infty}^{\infty} \frac{\begin{vmatrix} 0 & g_{\ell} & g_{\ell+m} & g_{\ell+m+k} \\ g_{-\ell} & 0 & g_m & g_{m+k} \\ g_{-\ell-m} & g_{-m} & 0 & g_k \\ g_{-\ell-m-k} & g_{-m-k} & g_{-k} & 0 \end{vmatrix}}{\left[\lambda-4t^2\right]\left[\lambda-4(t+k)^2\right]\left[\lambda-4(t+k+m)^2\right]\left[\lambda-4(t+k+m+\ell)^2\right]} .$$

The general law, according to which Δ_n can be expressed as an infinite sum involving n by n determinants, is apparent from (7.15) and (7.16). Now we have to remark briefly about the proof of Lemma 2 (Section 4). We know from (7.8) the general behavior of the coefficient c of a particular product of degree n in the g_ℓ. If we make ω purely imaginary and $\lambda = \omega^2$ negative, i.e. if we put

$$\omega = i\theta, \qquad \lambda = -\theta^2$$

then, for $\theta \to \infty$,

$$(7.17) \qquad c(\ell_1, \ldots, \ell_n)\left[\sin^2 \frac{\pi}{2} \omega\right]^{-1} = \mathcal{O}(\theta^{-d})$$

where d is the smaller of the differences between the degrees of the denominators and the numerators in $A(\omega)$ and $B(\omega)$. (Special attention must be given to the case where these two differences are equal and where d may be greater than either of these two differences because in $A \cos \pi\omega + B \sin \pi\omega$, the asymptotic behavior for $\omega = i\theta$, $\theta \to +\infty$ may be different

from that of A cos $\pi\omega$ and of A sin $\pi\omega$. This case has to be settled by putting $\omega = (1 + i)\theta$, and letting $\theta \to +\infty$). If we wish to compute d by using (7.17), we may apply the following.

Lemma 3: For $\theta \to +\infty$,

$$(7.18)\qquad \sum_{m_1,\dots,m_r=1}^{\infty} \sum_{t=-\infty}^{+\infty} \prod_{\nu=1}^{n} \left[\theta^2 + (t + m_1 + \dots + m_\nu)^2\right]^{-1} \cdot (\theta^2 + t^2)^{-1}$$

$$= \mathcal{O}(\theta^{r-2n-1}) .$$

The proof of (7.18) is based on a comparison of the multiple sum in (7.18) with the multiple integral

$$(7.19)\qquad \int_0^\infty \cdots \int_0^\infty d\mu_1 \dots d\mu_r \int_{-\infty}^{\infty} d\tau(\theta^2 + \tau^2)^{-1} \prod_{\nu=1}^{n} \left[\theta^2 + (\tau + \mu_1 + \dots + \mu_\nu)^2\right]^{-1} .$$

By substituting $\theta\tau, \theta\mu_1, \dots, \theta\mu_n$ for $\tau, \mu_1, \dots, \mu_n$, this integral is seen to be equal to

$$\theta^{r-2n-1} \int_0^\infty \cdots \int_0^\infty d\mu_1 \dots d\mu_r \int_{-\infty}^{\infty} d\tau(1 + \tau^2)^{-1} \prod_{\nu=1}^{n} \left[1+(\tau+\mu_1+\dots+\mu_\nu)^2\right]^{-1}$$

and here the integral is a constant independent of θ. On the other hand, it is easily seen that (7.19) is a majorant of the sum in (7.18).

Using Lemma 3, we can prove Lemma 2 as follows: Looking at the representation of Δ_n as a sum of determinants which are divided by certain products of the type appearing in (7.18), and picking a particular product

$$P = g_{\ell_1} \cdots g_{\ell_n}$$

appearing in these determinants, we see that P can appear in infinitely many of these determinants only if several of the subscripts $\ell_1, \ldots, \ell_n$ are coupled in pairs, say ℓ_1, ℓ_2 and ℓ_3, ℓ_4 etc. such that $\ell_1 = -\ell_2$, $\ell_3 = -\ell_4$ and so on. If we have r such pairs, the coefficient of P in Δ_n is essentially an $(r + 1)$-fold infinite sum of the type (7.18). But in a determinant of the type appearing in (7.16) (which illustrates the case $n = 4$) at most $[n/2]$ such pairs can appear, and this proves Lemma 2.

The corollary of Theorem 3, as stated in Section 4, is an immediate consequence of the fact that a skew symmetric determinant with an odd number of rows and columns vanishes indentically.

Next, we shall describe briefly a method for evaluating the sums involved in Δ_2, Δ_3 and Δ_4.

To compute Δ_2, we need the fact that

$$(7.20) \qquad S(\ell) = \sum_{t=-\infty}^{\infty} \left\{ \left[\omega^2 - 4t^2\right]\left[\omega^2 - 4(t+\ell)^2\right] \right\}^{-1}$$

$$= \frac{\pi \cos(\pi\omega/2)}{4\omega \sin(\pi\omega/2)} \; \frac{1}{\omega^2 - \ell^2}$$

for $\ell = \pm 1, \pm 2, \pm 3, \ldots$. This relation can be proved by comparing the formulas for Δ_2, arising from the methods (i) or (ii) described above, with the expression (7.14). But we can also prove (7.20) directly by using the formula

$$\sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{itx}}{\omega^2 - 4t^2} = \frac{\pi \cos(x\omega/2)}{2\omega \sin(\pi\omega/2)} ,$$

which can be proved by expanding cos $(x\omega/2)$ in a Fourier series in the interval $-\pi < x < \pi$. Now we have

$$\sum_{t=-\infty}^{+\infty} \frac{(-1)^t e^{itx}}{\omega^2 - 4t^2} = \sum_{t=-\infty}^{+\infty} \frac{(-1)^{t+\ell} e^{-i(t+\ell)x}}{\omega^2 - 4(t+\ell)^2} = (-1)^{\ell} e^{-i\ell x} \sum_{t=-\infty}^{+\infty} \frac{(-1)^t e^{-itx}}{\omega^2 - 4(t+\ell)^2}$$

$$= \frac{\pi \cos(x\omega/2)}{2\omega \sin(\omega\pi/2)}$$

and therefore

$$\sum_{t=-\infty}^{\infty} \frac{1}{\left[\lambda - 4t^2\right]\left[\lambda - 4(t+\ell)^2\right]} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{itx}}{\omega^2 - 4t^2} \sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{-itx}}{\omega^2 - 4(t+\ell)^2} \, dt$$

$$= \frac{\pi^2 (-1)^{\ell}}{8\pi\omega^2 \sin^2(\pi\omega/2)} \int_{-\pi}^{\pi} \cos^2 x\omega/2 \; e^{-i\ell x} \, dx = \frac{\pi \cos(\omega\pi/2)}{4\omega \sin(\omega\pi/2)\left[\omega^2 - \ell^2\right]}$$

which proves (7.20).

In order to calculate Δ_3, we have to evaluate the sum

$$(7.21) \qquad S(m,\ell) = \sum_{t=-\infty}^{\infty} \left\{\left[\lambda - 4t^2\right]\left[\lambda - 4(t+m)^2\right]\left[\lambda - 4(t+m+\ell)^2\right]\right\}^{-1}.$$

Using the sums $S(\ell)$ as defined and evaluated in (7.20), we have that $S(\ell)$ can also be written as

$$S(\ell) = \sum_{t=-\infty}^{+\infty} \left\{\left[\lambda - 4(t+m)\right]\left[\lambda - 4(t+m+\ell)^2\right]\right\}^{-1}$$

and therefore

$$S(m,\ell) = AS(m) + BS(\ell + m) + CS(\ell)$$

where

$$A = -\frac{1}{4\ell(m+\ell)}, \quad B = \frac{1}{4\ell m}, \quad C = -\frac{1}{4m(m+\ell)}$$

or

$$(7.22) \qquad S(m,\ell) = \frac{\pi\cos\pi\omega/2}{16\omega\sin\pi\omega/2} \quad \frac{3\lambda - \ell^2 - m^2 - \ell m}{(\lambda - \ell^2)(\lambda - m^2)\left[\lambda - (\ell + m)^2\right]}.$$

Obviously, (7.22) and (7.15) will give us the expression for $\Delta_3(\lambda)$ as stated in Theorem 2.

Finally, we have to evaluate formula (7.16) which gives us Δ_4. For this purpose, we must first write out the 4 by 4 determinants in the numerator; then we must collect all terms which involve the same product of the

g_n, and, finally, we must carry out whatever summations remain. This is a rather tedious job, and we shall not discuss it in detail. All we shall do here is to indicate which sums must be evaluated and how this evaluating can be done

Consider first

$$S(k,m,\ell) = \sum_{t=-\infty}^{\infty} \left\{\left[\lambda - 4t^2\right]\left[\lambda - 4(t+k)^2\right]\left[\lambda - 4(t+k+m)^2\right]\left[\lambda - 4(t+k+m+\ell)^2\right]\right\}^{-1} .$$

Using the method applied for the calculation of $S(m,\ell)$ in (7.22), we find that

$$(7.23) \qquad S(k,m,\ell) = -\frac{1}{4k(k+m)} S(m,\ell) + \frac{1}{4km} S(k+m,\ell) - \frac{1}{4m(k+m)} S(k,\ell+m) .$$

Since

$$\frac{1}{km} = \frac{1}{k(k+m)} + \frac{1}{m(k+m)} ,$$

we see that $S(k,m,\ell)$ tends toward zero like $\lambda^{-1}S(m,\ell)$ as $\lambda \to \infty$.

The main difficulty consists in computing the coefficients of $(g_\ell g_{-\ell})^2$ and of $g_m g_{-m} g_\ell g_{-\ell}$ in Δ_4. We shall use the following notations:

$$\tilde{S}(\ell) = \sum_{t=-\infty}^{\infty} \left\{\left[\lambda - 4t^2\right]\left[\lambda - 4(t+\ell)^2\right]\right\}^{-2}$$

$$S^*(\ell) = \sum_{t=-\infty}^{\infty} \left[\lambda - 4t^2\right]^{-2} \left\{\left[\lambda - 4(t-\ell)^2\right]\left[\lambda - 4(t+\ell)^2\right]\right\}^{-1}$$

$$\tilde{S}(\ell,m) = \sum_{t=-\infty}^{\infty} \left[\lambda - 4t^2\right]^{-2} \left[\lambda - 4(t+\ell)^2\right]^{-1} \left[\lambda - 4(t+m)^2\right]^{-1}$$

$$S^*(\ell,m) = \sum_{t=-\infty}^{+\infty} \left\{\left[\lambda - 4t^2\right]\left[\lambda - 4(t+\ell)^2\right]\left[\lambda - 4(t+m)^2\right]\left[\lambda - 4(t+\ell+m)^2\right]\right\}^{-1}.$$

Then the coefficient of $(g_\ell g_{-\ell})^2$ on the right-hand side of (7.16), for a fixed ℓ, is

$$\frac{1}{2}\left[S(\ell)\right]^2 - \frac{1}{2}\tilde{S}(\ell) = -S^*_\ell \quad . \tag{7.24}$$

The coefficient of $g_\ell g_{-\ell} g_m g_{-m}$, for fixed values of ℓ,m and $\ell > m$ is

$$\left[S(\ell)\right]^2 - 2\tilde{S}(\ell,m) - S(-\ell,m) - S(\ell,-m) + 2S^*(\ell,m) \quad . \tag{7.25}$$

We observe now that it follows from the formulas after (7.20) that

$$\frac{\partial}{\partial\lambda} \sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{itx}}{\lambda - 4t^2} = - \sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{itx}}{(\lambda-4t^2)^2} = \frac{\partial}{\partial\lambda} \frac{\pi \cos(x\sqrt{\lambda}/2)}{2\sqrt{\lambda}\sin(\pi\sqrt{\lambda}/2)} \tag{7.26}$$

By applying the same method to (7.26) which we used in proving (7.20), we find that

$$\sum_{t=-\infty}^{\infty} \left\{\left[\lambda - 4t^2\right]\left[\lambda - 4(t-\ell)^2\right]\right\}^{-1} = \tilde{S}(\ell)$$

$$= \frac{(-1)^\ell}{2\pi} \frac{\pi^2}{64\lambda^2 \sin^2(\pi\sqrt{\lambda}/2)} \int_{-\pi}^{\pi} e^{i\ell x} \left[\frac{2\cos(x\sqrt{\lambda}/2)}{\sqrt{\lambda}} + x\sin(x\sqrt{\lambda}/2)\right.$$

$$+ \frac{\cos(x\sqrt{\lambda}/2)\ \cos(\pi\sqrt{\lambda}/2)}{\sin(\pi\sqrt{\lambda}/2)} \Bigg]^2 dx$$

$$= \frac{\pi}{128\lambda^2 \sin^2(\pi\sqrt{\lambda}/2)} \left\{ \frac{\sin\pi\sqrt{\lambda}}{\sqrt{\lambda}} \ \frac{10\lambda^2 - 2\lambda\ell^2}{(\lambda-\ell^2)^3} + 2\pi \ \frac{\lambda^2 + \lambda\ell^2}{\ell^2(\lambda-\ell^2)^2} \right\} .$$

By the same type of argument, we find that

$$S'(\ell) = \sum_{t=-\infty}^{\infty} \left[\lambda - 4t^2\right]^{-2} \left[\lambda - 4(t-\ell)^2\right]^{-1}$$

$$= \frac{\pi}{32\lambda^{3/2} \sin^2(\pi\sqrt{\lambda}/2)} \left\{ \sin\pi\sqrt{\lambda} \ \frac{3\lambda - \ell^2}{(\lambda-\ell^2)^2} + \frac{\pi\sqrt{\lambda}}{\lambda - \ell^2} \right\}$$

and

$$8\ell^2 S^*(\ell) = -\ S'(\ell) - S'(-\ell) + 2S(\ell, -\ell)$$

where

$$S(\ell, -\ell) = S(\ell, \ell) .$$

Finally, we find that

$$4\ell m(\ell-m)\tilde{S}(-\ell,-m) = 4\ell m(\ell-m)\tilde{S}(\ell,m) = \ell S'(\ell) - mS'(m) - (\ell-m)S(m,m-\ell)$$

and these formulas permit us to compute all of the sums occurring in the evaluation of Δ_4. It should be noted that, in writing down these formulas, we have frequently used

$$\sum_{t=-\infty}^{\infty} = \sum_{t=\infty}^{-\infty} = \sum_{\tau=-\infty}^{\infty}$$

where $\tau = - t$.

This concludes the proof of Theorem 2.

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